CS 188: Artificial Intelligence
Spring 2010

Lecture 9: MDPs

2/16/2010

Pieter Abbeel – UC Berkeley

Many slides adapted from Dan Klein

Announcements

- Assignments
  - P2 due Thursday
  - We reserved Soda 271 on Wednesday Feb 17 from 4 to 6. One of the GSI's will periodically drop in to see if he can provide any clarifications/assistance. It's a great opportunity to meet other students who might still be looking for a partner.
- Readings:
  - For MDPs / reinforcement learning, we're using an online reading
  - Different treatment and notation than the R&N book, beware!
  - Lecture version is the standard for this class

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Example: Insurance

- Consider the lottery [0.5,$1000; 0.5,$0]
  - What is its expected monetary value (EMV)? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!

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Example: Insurance

- Because people ascribe different utilities to different amounts of money, insurance agreements can increase both parties' expected utility

You own a car. Your lottery:
L_Y = [0.8, $0 ; 0.2, -$200]
i.e., 20% chance of crashing

You do not want -$200!

$U_Y(L_Y) = 0.2 * U_Y(-\$200) = -200$
$U_Y(-\$50) = -150$

Amount	Your Utility U_Y
$0	0
-$50	-150
-$200	-1000

Example: Insurance

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Insurance company buys risk:
L_I = [0.8, \$50 ; 0.2, -\$150]
i.e., \$50 revenue + your L_Y

Insurer is risk-neutral:
U(L)=U(EMV(L))

$U_I(L_I) = U(0.8*50 + 0.2*(-150))$
$= U(\$10) > U(\$0)$

Example: Human Rationality?

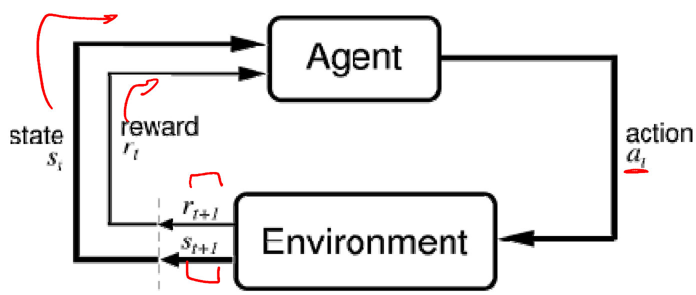
- Famous example of Allais (1953)
  - A: [0.8,\$4k; 0.2,\$0] 5
  - B: [1.0,\$3k; 0.0,\$0] ≥30

  - C: [0.2,\$4k; 0.8,\$0] ≥30
  - D: [0.25,\$3k; 0.75,\$0] 4

- Most people prefer B > A, C > D
- But if U(\$0) = 0, then
  - B > A ⇒ U(\$3k) > 0.8 U(\$4k)
  - C > D ⇒ 0.2 U(\$4k) > 0.25 U(\$3k)
    equivalently: 0.8 U(\$4k) > U(\$3k) ?

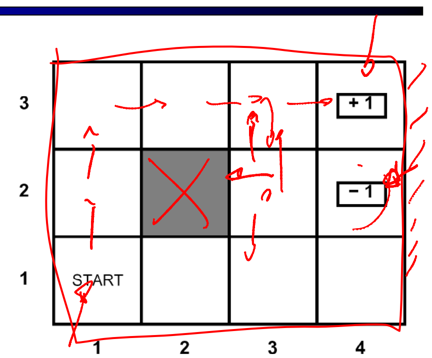
Reinforcement Learning

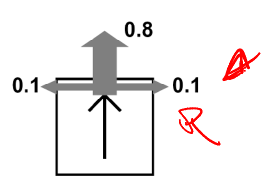
- Basic idea:
  - Receive feedback in the form of rewards
  - Agent's utility is defined by the reward function
  - Must learn to act so as to maximize expected rewards



Grid World

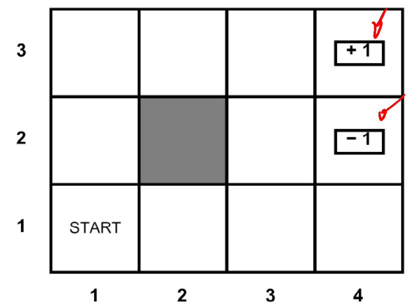
- The agent lives in a grid
- Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Small "living" reward each step
- Big rewards come at the end
- Goal: maximize sum of rewards

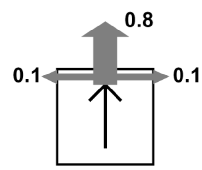




Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state
- MDPs are a family of non-deterministic search problems
  - Reinforcement learning: MDPs where we don't know the transition or reward functions





What is Markov about MDPs?

- Andrey Markov (1856-1922)
- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means:

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

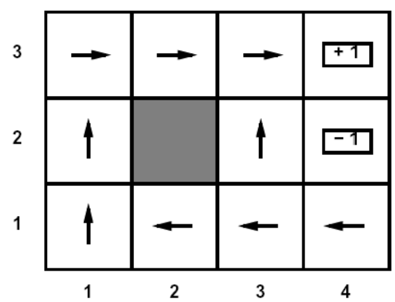
$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

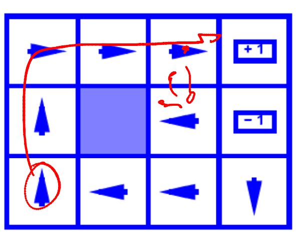
Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal
- In an MDP, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy maximizes expected utility if followed
  - Defines a reflex agent

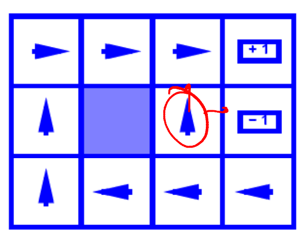
Optimal policy when R(s, a, s') = -0.03 for all non-terminals s



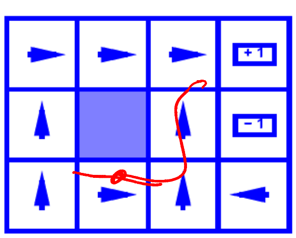
Example Optimal Policies



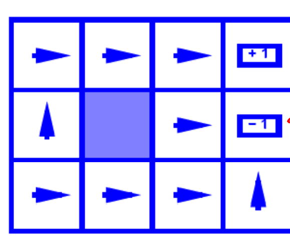
R(s) = -0.01



R(s) = -0.03



R(s) = -0.4



R(s) = -2.0

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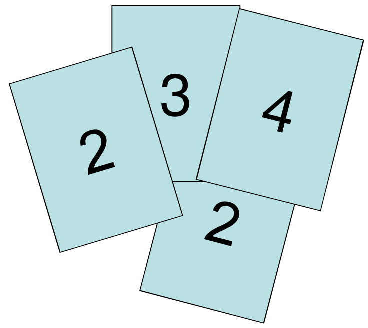
Example: High-Low

- Three card types: 2, 3, 4
- Infinite deck, twice as many 2's
- Start with 3 showing
- After each card, you say "high" or "low"
- New card is flipped
- If you're right, you win the points shown on the new card
- Ties are no-ops
- If you're wrong, game ends



- Differences from expectimax:
  - #1: get rewards as you go
  - #2: you might play forever!

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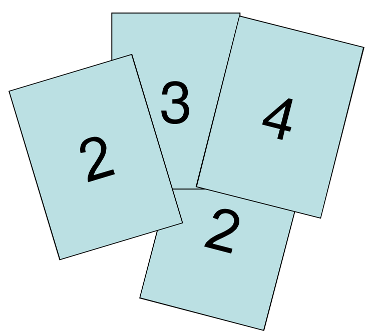
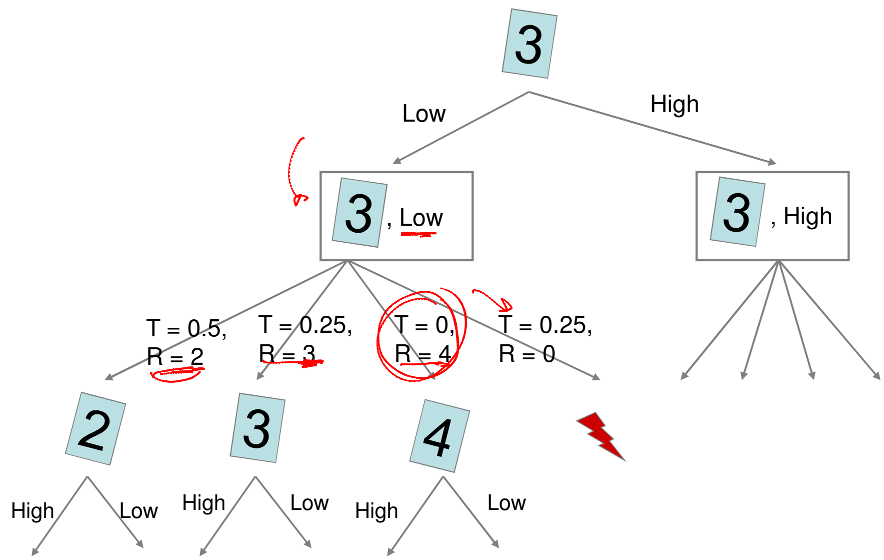
High-Low as an MDP

- States: 2, 3, 4, done
- Actions: High, Low
- Model: T(s, a, s'):
  - P(s'=4 | 4, Low) = 1/4
  - P(s'=3 | 4, Low) = 1/4
  - P(s'=2 | 4, Low) = 1/2
  - P(s'=done | 4, Low) = 0
  - P(s'=4 | 4, High) = 1/4
  - P(s'=3 | 4, High) = 0
  - P(s'=2 | 4, High) = 0
  - P(s'=done | 4, High) = 3/4
  - …



- Rewards: R(s, a, s'):
  - Number shown on s' if s ≠ s' and a is "correct"
  - 0 otherwise
- Start: 3

Example: High-Low



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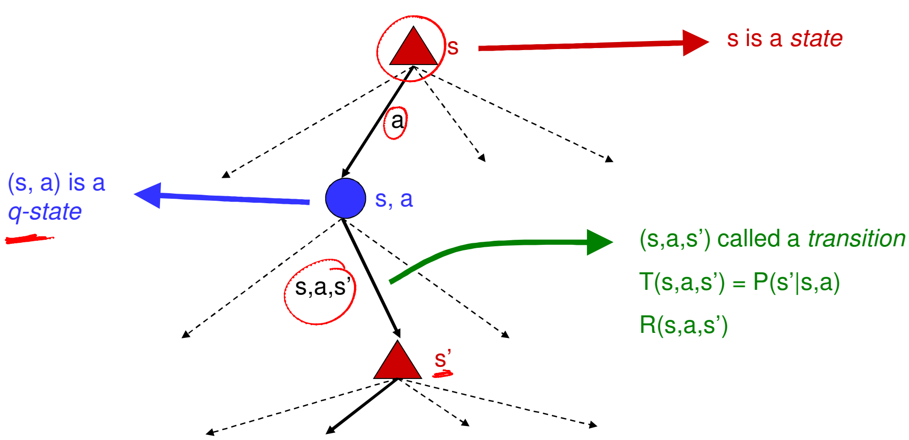
MDP Search Trees

- Each MDP state gives an expectimax-like search tree



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Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$[r_1, r_2, \ldots] \succ [r_1', r_2', \ldots]$

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r_0', r_1', r_2', \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r_0', r_1', r_2', \ldots]$$

- Theorem: only two ways to define stationary utilities
  - Additive utility:

$$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$

  - Discounted utility:

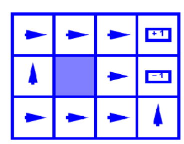
$$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

$\sum_{i=0}^{\infty} \gamma^i = \frac{1}{1-\gamma}$

$\gamma < 1$

Infinite Utilities?!

- Problem: infinite state sequences have infinite rewards
- Solutions:
  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "done" for High-Low)
  - Discounting: for 0 < γ < 1

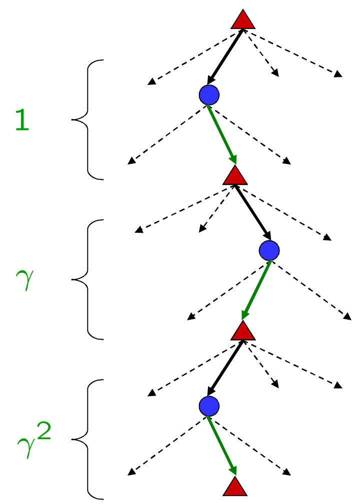


$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\text{max}}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus

Discounting

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



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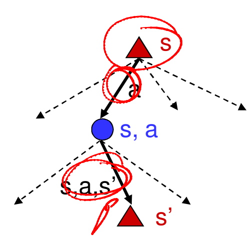
Recap: Defining MDPs

- Markov decision processes:
  - States S
  - Start state s_0
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)



- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards

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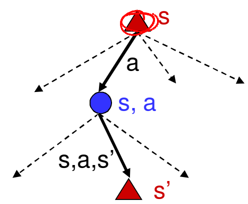
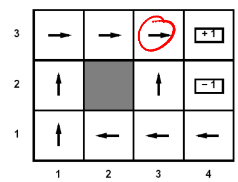
Optimal Utilities

- Fundamental operation: compute the values (optimal expectimax utilities) of states s
- Why? Optimal values define optimal policies!
- Define the value of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- Define the value of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally
- Define the optimal policy:
  $\pi^*(s)$ = optimal action from state s



$V^*(3,3)$

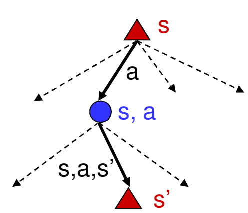
	1	2	3	4
3	0.812	0.868	0.912	+1
2	0.762		0.660	−1
1	0.705	0.655	0.611	0.388



The Bellman Equations

- Definition of “optimal utility” leads to a simple one-step lookahead relationship amongst optimal utility values:

  Optimal rewards = maximize over first action and then follow optimal policy



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

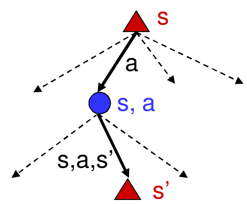
Solving MDPs

- We want to find the optimal policy π^*
- Proposal 1: modified expectimax search, starting from each state s:

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a Q^*(s,a)$$



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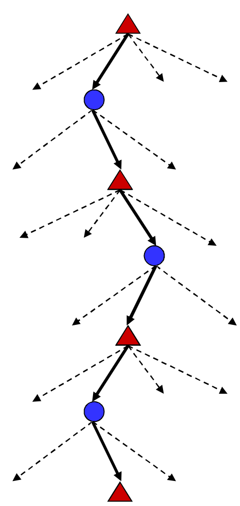
Why Not Search Trees?

- Why not solve with expectimax?
- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)
- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!



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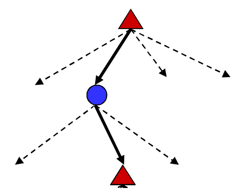
Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - As $k \to \infty$, it approaches the optimal value
  - Why:
    - If discounting, distant rewards become negligible
    - If terminal states reachable from everywhere, fraction of episodes not ending becomes negligible
    - Otherwise, can get infinite expected utility and then this approach actually won't work



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